Second-Order Differential Subordination for Analytic Functions with Fixed Initial Coefficient

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Abstract. Functions with fixed initial coefficient have been widely studied. A new methodology is proposed in this paper by making appropriate modifications and improvements to the theory of second-order differential subordination. Several interesting examples are given. The results obtained are applied to the classes of convex and starlike functions with fixed second coefficient.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For a fixed positive integer n, let $\mathcal{H}[a, n]$ be its subset consisting of functions p of the form

$$p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots .$$

The familiar subclass $\mathcal{S}$ of $\mathcal{H}[0, 1]$ consists of normalized univalent functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in $\mathbb{D}$. It is a remarkable fact that the second coefficient plays an important role in univalent function theory; indeed it influences growth and distortion estimates [10] for functions in the class $\mathcal{S}$. For this reason, there is a continued interest in the investigations of how the second coefficient shaped the geometric properties of important subclasses of functions. Works in this direction include those of [6, 7, 11, 12, 16, 17].

This paper studies further analytic functions with fixed initial coefficient. The methodology used here will be differential subordination, applied from making appropriate modifications and improvements to its existing theory. Many of the significant works on differential subordination have been pioneered by Miller and Mocanu [13, 14], and their monograph [15] compiled their great efforts over more than two decades. In recent years, various authors have successfully applied the theory of

differential subordination to address many important problems in the field. These works include those of [1–5, 8, 9, 18].

To state the central theme of Miller and Mocanu's theory on differential subordination, let p be an analytic function in $\mathbb{D}$ and $\psi(r, s, t)$ be a complex function defined in a domain of $\mathbb{C}^3$. Consider a class of functions Ψ, and two subsets Ω and Δ in $\mathbb{C}$. Given any two quantities, the aim of the theory of second-order differential subordination is to determine the third so that the following differential implication is satisfied:

$$\psi \in \Psi \quad \text{and} \quad \{\psi(p(z), zp'(z), z^2 p''(z)) : z \in \mathbb{D}\} \subset \Omega \quad \Rightarrow \quad p(\mathbb{D}) \subset \Delta.$$

Let $\mathcal{H}_\beta[a, n]$ consist of analytic functions p of the form

$$p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots,$$

where the coefficient p_n is a fixed constant β in $\mathbb{C}$. Without loss of generality, β is assumed to be a nonnegative real number.

Let $\psi : \mathbb{C}^3 \to \mathbb{C}$ be analytic in a domain D and let h be univalent in $\mathbb{D}$. Suppose $p \in \mathcal{H}_\beta[a, n]$, $(p(z), zp'(z), z^2 p''(z)) \in D$ when $z \in \mathbb{D}$, and p satisfies the second-order differential subordination

$$(1.1) \qquad\qquad \psi(p(z), zp'(z), z^2 p''(z)) \prec h(z).$$

Then p is called a β-*solution* of the differential subordination. The univalent function q is said to be a β-*dominant* of the differential subordination (1.1) if $p \prec q$ for all p satisfying (1.1). If $\widetilde{q}$ is a β-dominant of (1.1) and $\widetilde{q} \prec q$ for all β-dominants q of (1.1), then $\widetilde{q}$ is called the β-*best dominant* of (1.1). The class of functions Ψ_β such that (1.1) is satisfied for $\psi \in \Psi_\beta$ is called the class of β-*admissible functions*.

Section 2 of this paper deals with the basic lemmas for functions with fixed initial coefficient. In Section 3 a suitable class $\Psi_{n,\beta}(\Omega, q)$ of β-admissible functions is defined and theorems analogous to those of Miller and Mocanu [15] are obtained. These results are applied to two important particular cases corresponding to $q(\mathbb{D})$ being a disk or a half-plane. Examples of differential inequalities and subordinations are presented in Section 4. The paper concludes with interesting applications of the newly formulated theory to the classes of normalized convex and starlike univalent functions with fixed second coefficient.

The following extended version of Schwarz Lemma (see [11]) is required in our investigations.

Lemma 1.1 (Extended Schwarz Lemma). *Let $w(z) = a_1 z + \cdots$ be an analytic map of the unit disk $\mathbb{D}$ into itself. Then $|a_1| \le 1$, and*

$$|w(z)| \le \frac{|z|(|z| + |a_1|)}{1 + |a_1||z|}.$$

Equality holds at some point $z \ne 0$ if and only if

$$w(z) = \frac{e^{-it} z(z + a_1 e^{it})}{1 + \bar{a}_1 e^{-it} z}, \quad t \ge 0.$$

2. Fundamental lemmas for functions with fixed initial coefficient

Let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. In this section, we will prove several basic lemmas for functions with fixed initial coefficient.

Lemma 2.1. *Let* $z_0 = r_0 e^{i\theta_0}$, $(r_0 < 1)$, *and* $g(z) = g_n z^n + g_{n+1} z^{n+1} + \cdots$ *be continuous in* $\overline{\mathbb{D}}_{r_0}$, *analytic in* $\mathbb{D}_{r_0} \cup \{z_0\}$ *with* $g(z) \not\equiv 0$, *and* $n \geq 1$. *If*

$$|g(z_0)| = \max_{|z| \leq r_0} |g(z)|$$

then

$$\frac{z_0 g'(z_0)}{g(z_0)} = m \quad and \quad \mathrm{Re}\left(\frac{z_0 g''(z_0)}{g'(z_0)} + 1\right) \geq m,$$

where

(2.1)
$$m \geq n + \frac{|g(z_0)| - |g_n| r_0^n}{|g(z_0)| + |g_n| r_0^n}.$$

Proof. The first two assertions follow from Lemma 2.2a in [15, p. 19]. It remains to prove (2.1). Let $h : \overline{\mathbb{D}} \to \mathbb{C}$ be defined by

$$h(z) = \frac{g(z_0 z)}{g(z_0) z^{n-1}} = s_n z + \cdots, \quad \text{where} \quad s_n = \frac{g_n z_0^n}{g(z_0)}.$$

Then h is continuous in $\overline{\mathbb{D}}$ and analytic in $\mathbb{D} \cup \{1\}$, and the maximum principle readily gives

$$|h(z)| \leq \max_{|z|=1} |h(z)| = \frac{1}{|g(z_0)|} \max_{|z|=1} |g(z_0 z)| = 1.$$

Since $h(0) = 0$, the extended Schwarz Lemma (Lemma 1.1) yields $|s_n| \leq 1$, and

$$\left|\frac{g(z_0 z)}{g(z_0) z^{n-1}}\right| = |h(z)| \leq \frac{|z|(|z| + |s_n|)}{1 + |s_n||z|}.$$

In particular, at the point $z = r$, $0 \leq r < 1$,

(2.2)
$$\mathrm{Re}\, \frac{g(z_0 r)}{g(z_0)} \leq \left|\frac{g(z_0 r)}{g(z_0)}\right| \leq \frac{r^n(r + |s_n|)}{1 + |s_n| r}.$$

Since $m = z_0 g'(z_0)/g(z_0)$,

$$m = \frac{d}{dr}\left(\frac{g(z_0 r)}{g(z_0)}\right)\bigg|_{r=1} = \lim_{r\to 1} \frac{g(z_0 r) - g(z_0)}{(r-1)z_0} \frac{z_0}{g(z_0)} = \lim_{r\to 1}\left(1 - \frac{g(z_0 r)}{g(z_0)}\right)\frac{1}{1-r}.$$

Taking real parts and using (2.2), it follows that

$$m = \lim_{r\to 1}\left(1 - \mathrm{Re}\, \frac{g(z_0 r)}{g(z_0)}\right)\frac{1}{1-r}$$

$$\geq \lim_{r\to 1}\left(1 - \frac{r^n(r + |s_n|)}{1 + |s_n| r}\right)\frac{1}{1-r}$$

$$= n + \frac{1 - |s_n|}{1 + |s_n|}$$

$$= n + \frac{|g(z_0)| - |g_n| r_0^n}{|g(z_0)| + |g_n| r_0^n}. \qquad \blacksquare$$

To state the second technical lemma, let Q be the class of functions q that are analytic and injective in $\overline{\mathbb{D}} \setminus E(q)$, where

$$E(q) := \left\{ \zeta \in \partial\mathbb{D} : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$.

Lemma 2.2. *Let* $q \in Q$ *with* $q(0) = a$, *and* $p \in \mathcal{H}_\beta[a, n]$ *with* $p(z) \not\equiv a$. *If there exists a point* $z_0 \in \mathbb{D}$ *such that* $p(z_0) \in q(\partial\mathbb{D})$ *and* $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{D})$, *then*

$$(2.3) \qquad\qquad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$$

and

$$(2.4) \qquad\qquad \operatorname{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \operatorname{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right),$$

where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ *and*

$$(2.5) \qquad\qquad m \geq n + \frac{|q'(0)| - \beta|z_0|^n}{|q'(0)| + \beta|z_0|^n}.$$

Proof. Except for (2.5), the assertions here follow from Lemma 2.2d in [15, p. 24]. Thus, we shall only prove (2.5). Let g be defined by

$$g(z) = q^{-1}(p(z)), \quad |z| \leq |z_0|.$$

Then g is analytic in $\{z : |z| \leq |z_0|\}$ and satisfies $|g(z_0)| = 1$, $g(0) = 0$, $|g(z)| \leq 1$ for $|z| \leq |z_0|$ and

$$g^{(k)}(0) = p^{(k)}(0) = 0 \quad \text{for} \quad k = 1, 2, \cdots, n - 1.$$

If $g(z) = g_n z^n + g_{n+1} z^{n+1} + \cdots$, then the relation $q(g(z)) = p(z)$ gives $g_n = \beta/q'(0)$. Lemma 2.1 and Lemma 2.2d in [15, p. 24] yield an m satisfying (2.3) and (2.4) where

$$m \geq n + \frac{|g(z_0)| - |g_n||z_0|^n}{|g(z_0)| + |g_n||z_0|^n}.$$

Since $|g(z_0)| = 1$, it follows that

$$m \geq n + \frac{1 - |g_n||z_0|^n}{1 + |g_n||z_0|^n} = n + \frac{|q'(0)| - \beta|z_0|^n}{|q'(0)| + \beta|z_0|^n}. \qquad\blacksquare$$

Following Miller and Mocanu [15], two important functions of $q \in Q$ will be considered, namely, when $q(\mathbb{D})$ is a disk or a half-plane.

Case 1. The set $q(\mathbb{D})$ is the disk $\Delta = \mathbb{D}_M = \{w : |w| < M\}$. Here the function

$$q(z) = M \frac{Mz + a}{M + \bar{a}z} \quad (z \in \mathbb{D}),$$

with $M > 0$ and $|a| < M$, is univalent in $\overline{\mathbb{D}}$ and satisfies $q(\mathbb{D}) = \Delta$, $q(0) = a$ and $q \in Q$.

Case 2. The set $q(\mathbb{D})$ is the half-plane $\Delta = \{w : \operatorname{Re} w > \alpha\}$. Then

$$q(z) = \frac{a - (2\alpha - \bar{a})z}{1 - z} \quad (z \in \mathbb{D}),$$

where $\alpha \in \mathbb{R}$ and $\operatorname{Re} a > \alpha$, is univalent in $\overline{\mathbb{D}} \setminus \{1\}$ and satisfies $q(\mathbb{D}) = \Delta$, $q(0) = a$ and $q \in Q$.

The following two lemmas are important in the above cases.

Lemma 2.3. *Let $p \in \mathcal{H}_\beta[a, n]$, $\beta \neq 0$. If $z_0 \in \mathbb{D}$ and*

$$|p(z_0)| = \max_{|z| \leq |z_0|} |p(z)|,$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} \geq \left(n + \frac{|p(z_0)|^2 - |a|^2 - \beta|p(z_0)||z_0|^n}{|p(z_0)|^2 - |a|^2 + \beta|p(z_0)||z_0|^n} \right) \frac{|p(z_0) - a|^2}{|p(z_0)|^2 - |a|^2}$$

and

$$\operatorname{Re}\left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq \left(n + \frac{|p(z_0)|^2 - |a|^2 - \beta|p(z_0)||z_0|^n}{|p(z_0)|^2 - |a|^2 + \beta|p(z_0)||z_0|^n} \right) \frac{|p(z_0) - a|^2}{|p(z_0)|^2 - |a|^2}.$$

Proof. Set $M = |p(z_0)|$. Similar to the proof of Lemma 2.2e in [15, p. 25], Lemma 2.2 shows there exists an m satisfying (2.5) such that
(2.6)

$$z_0 p'(z_0) = m p(z_0) \frac{|p(z_0) - a|^2}{|p(z_0)|^2 - |a|^2} \quad \text{and} \quad \operatorname{Re}\left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq m \frac{|p(z_0) - a|^2}{|p(z_0)|^2 - |a|^2}.$$

Since

$$q'(z) = M \frac{M^2 - |a|^2}{(M + \bar{a}z)^2},$$

it readily follows that

$$q'(0) = \frac{|p(z_0)|^2 - |a|^2}{|p(z_0)|}.$$

Therefore (2.5) becomes

$$m \geq n + \frac{|p(z_0)|^2 - |a|^2 - \beta|p(z_0)||z_0|^n}{|p(z_0)|^2 - |a|^2 + \beta|p(z_0)||z_0|^n},$$

so that (2.6) gives the desired result. ∎

Remark 2.1. If $a = 0$, then Lemma 2.3 reduces to Lemma 2.1.

Lemma 2.4. *Let $p \in \mathcal{H}_\beta[a, n]$, $\beta \neq 0$. If $z_0 \in \mathbb{D}$ and*

$$\operatorname{Re} p(z_0) = \min_{|z| \leq |z_0|} \operatorname{Re} p(z),$$

then

$$z_0 p'(z_0) \leq -\frac{1}{2}\left(n + \frac{2|\operatorname{Re}(a - p(z_0))| - \beta|z_0|^n}{2|\operatorname{Re}(a - p(z_0))| + \beta|z_0|^n} \right) \frac{|p(z_0) - a|^2}{\operatorname{Re}(a - p(z_0))}$$

and

$$\operatorname{Re}\left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq 0.$$

Proof. If we set $\alpha = \operatorname{Re} p(z_0)$, then Lemma 2.2f in [15, p. 26] and Lemma 2.2 give the existence of m satisfying (2.5) such that

(2.7) $$z_0 p'(z_0) = -\frac{m}{2} \frac{|p(z_0) - a|^2}{\operatorname{Re}[a - p(z_0)]} \quad \text{and} \quad \operatorname{Re}\left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1\right) \geq 0.$$

Now

$$q'(z) = \frac{2(\operatorname{Re} a - \alpha)}{(1 - z)^2}$$

implies

$$q'(0) = 2(\operatorname{Re} a - \alpha) = 2\operatorname{Re}(a - p(z_0)).$$

Therefore (2.5) becomes

$$m \geq n + \frac{2|\operatorname{Re}(a - p(z_0))| - \beta|z_0|^n}{2|\operatorname{Re}(a - p(z_0))| + \beta|z_0|^n},$$

so that (2.7) gives the desired result. ∎

3. β-admissible functions and fundamental theorems

In this section we will prove fundamental results related to the implication

$$\{\psi(p(z), zp'(z), z^2 p''(z)) : z \in \mathbb{D}\} \subset \Omega \quad \Rightarrow \quad p(\mathbb{D}) \subset \Delta$$

for a suitably defined class of functions.

Definition 3.1 (β-Admissibility Condition). *Let Ω be a domain in $\mathbb{C}$, $q \in Q$, and $\beta \in \mathbb{C}$ with $\beta \leq |q'(0)|$. The class $\Psi_{n,\beta}(\Omega, q)$ consists of β-admissible functions $\psi : \mathbb{C}^3 \to \mathbb{C}$ satisfying the following conditions:*

(i) *$\psi(r, s, t)$ is continuous in a domain $D \subset \mathbb{C}^3$,*

(ii) *$(q(0), 0, 0) \in D$ and $\psi(q(0), 0, 0) \in \Omega$,*

(iii) *$\psi(r_0, s_0, t_0) \notin \Omega$ whenever $(r_0, s_0, t_0) \in D$, $r_0 = q(\zeta)$, $s_0 = m\zeta q'(\zeta)$ and*

$$\operatorname{Re}\left(\frac{t_0}{s_0} + 1\right) \geq m\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

where $|\zeta| = 1$, $q(\zeta)$ is finite and

$$m \geq n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}.$$

The class $\Psi_{1,\beta}(\Omega, q)$ is denoted by $\Psi_{\beta}(\Omega, q)$.

Note that Ω is not required to be simply-connected or has a particularly nice boundary as we do for $q(\mathbb{D})$. If $\beta = |q'(0)|$, then the concept of β-admissibility coincides with the usual admissibility as discussed in [14], that is, $\Psi_n \equiv \Psi_{n,|q'(0)|}$. It is also evident from the definition that

$$\Psi_n \equiv \Psi_{n,|q'(0)|} \subseteq \Psi_{n,\beta_1} \subseteq \Psi_{n,\beta_2} \subseteq \Psi_{n,0} \equiv \Psi_{n+1} \quad (0 \leq \beta_2 \leq \beta_1 \leq |q'(0)|).$$

In view of the above inclusions, it is assumed throughout this sequel that $0 < \beta \leq |q'(0)|$.

Theorem 3.1. *Let* $q(0) = a$, $\psi \in \Psi_{n,\beta}(\Omega, q)$ *with associated domain* D, *and* $\beta \in \mathbb{C}$ *with* $0 < \beta \le |q'(0)|$. *Let* $p \in \mathcal{H}_\beta[a, n]$. *If* $(p(z), zp'(z), z^2 p''(z)) \in D$ *and*

$$(3.1) \qquad \psi(p(z), zp'(z), z^2 p''(z)) \in \Omega \quad (z \in \mathbb{D}),$$

then $p \prec q$.

Proof. Taking into account that q is univalent in $\mathbb{D}$ and $p(0) = q(0) = a$, it remains to show that $p(\mathbb{D}) \subset q(\mathbb{D})$. Let, if possible, $p(\mathbb{D}) \not\subset q(\mathbb{D})$. Then there exists a point $z_0 \in \mathbb{D}$ for which $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{D})$ and $p(\{z : |z| \le |z_0|\}) \not\subset q(\mathbb{D})$. Since $p(\{z : |z| \le |z_0|\}) \subset q(\overline{\mathbb{D}})$, therefore $p(z_0) \in \partial q(\mathbb{D}) = q(\partial\mathbb{D})$. At the point z_0, Lemma 2.2 shows that

$$p(z_0) = q(\zeta_0), \quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$$

and

$$\mathrm{Re}\left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1\right) \ge m\,\mathrm{Re}\left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right),$$

where $|\zeta_0| = 1$, $q(\zeta_0)$ is finite and

$$m \ge n + \frac{|q'(0)| - \beta|z_0|^n}{|q'(0)| + \beta|z_0|^n}.$$

The function $H : [0, 1] \to \mathbb{R}$ given by

$$H(r) = \frac{a - br^n}{a + br^n}, \quad a \ge 0, b \ge 0$$

is a decreasing function of r, and so

$$m \ge n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}.$$

With $r_0 = p(z_0)$, $s_0 = z_0 p'(z_0)$ and $t_0 = z_0^2 p''(z_0)$ in part (iii) of Definition 3.1, the condition

$$\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z)) \notin \Omega$$

contradicts (3.1). Hence $p(\mathbb{D}) \subset q(\mathbb{D})$ and $p \prec q$. ∎

The proof of the following result is similar to Corollary 1.1 in [14, p. 160].

Corollary 3.1. *Let* q *be univalent in* $\mathbb{D}$ *with* $q(0) = a$, *and* $q_\rho(z) = q(\rho z)$, $0 < \rho < 1$. *Let* $\psi \in \Psi_{n,\beta}(\Omega, q_\rho)$ *with domain* D, $0 < \rho < 1$, *where* $\beta \in \mathbb{C}$ *with* $0 < \beta \le |q'(0)|$, *and* $p \in \mathcal{H}_\beta[a, n]$. *If* $(p(z), zp'(z), z^2 p''(z)) \in D$ *and*

$$\psi(p(z), zp'(z), z^2 p''(z)) \in \Omega \quad (z \in \mathbb{D}),$$

then $p \prec q$.

Definition 3.2. *Let* $\Omega \ne \mathbb{C}$ *be a simply connected domain in* $\mathbb{C}$, $q \in Q$ *and* $\beta \in \mathbb{C}$. *Let* h *be a conformal mapping of* $\mathbb{D}$ *onto* Ω. *Denote by* $\Psi_{n,\beta}(h, q)$ *the class of functions* $\psi \in \Psi_{n,\beta}(\Omega, q) = \Psi_{n,\beta}(h(\mathbb{D}), q)$ *which are analytic in their associated domains* D *and satisfy* $\psi(q(0), 0, 0) = h(0)$. *We write* $\Psi_{1,\beta}(h, q)$ *as* $\Psi_\beta(h, q)$.

The following theorem and corollary are immediate consequences of Theorem 3.1 and Corollary 3.1.

Theorem 3.2. *Let $q(0) = a$ and $\psi \in \Psi_{n,\beta}(h, q)$ with associated domain D, where $\beta \in \mathbb{C}$ with $0 < \beta \leq |q'(0)|$. Let $p \in \mathcal{H}_\beta[a, n]$. If $(p(z), zp'(z), z^2 p''(z)) \in D$ and*

$$\psi(p(z), zp'(z), z^2 p''(z)) \prec h(z) \quad (z \in \mathbb{D}),$$

then $p \prec q$.

Corollary 3.2. *Let h and q be univalent in $\mathbb{D}$ with $q(0) = a$, and let $h_\rho(z) = h(\rho z)$, $q_\rho(z) = q(\rho z)$, for $0 < \rho < 1$. Let $\psi \in \Psi_{n,\beta}(h_\rho, q_\rho)$ with domain D, for $0 < \rho < 1$, where $\beta \in \mathbb{C}$ with $0 < \beta \leq |q'(0)|$ and let $p \in \mathcal{H}_\beta[a, n]$. If $(p(z), zp'(z), z^2 p''(z)) \in D$ and*

$$\psi(p(z), zp'(z), z^2 p''(z)) \prec h(z) \quad (z \in \mathbb{D}),$$

then $p \prec q$.

3.1. Two special cases

Let us next formulate the theorems above to the two important examples of $q(\mathbb{D})$ being a disk and $q(\mathbb{D})$ being a half-plane considered earlier.

Case 1. The disk $\Delta = \mathbb{D}_M = \{w : |w| < M\}$. Here the function

$$q(z) = M \frac{Mz + a}{M + \bar{a}z} \quad (z \in \mathbb{D}),$$

where $M > 0$ and $|a| < M$, is univalent in $\overline{\mathbb{D}}$ and satisfies $q(\mathbb{D}) = \Delta$, $q(0) = a$ and $q \in Q$. To describe the class of β-admissible functions in this case, set $\Psi_{n,\beta}(\Omega, M, a) := \Psi_{n,\beta}(\Omega, q)$ and when $\Omega = \Delta$, denote the class by $\Psi_{n,\beta}(M, a)$. Comparing with Lemma 2.2e in [15, p. 25], the condition of β-admissibility becomes

$$\psi(r, s, t) \notin \Omega \quad \text{whenever} \quad (r, s, t) \in D,$$

$$r = q(\zeta) = M e^{i\theta},$$

$$(3.2) \qquad\qquad s = m\zeta q'(\zeta) = m \frac{M|M - \bar{a}e^{i\theta}|^2}{M^2 - |a|^2} e^{i\theta}, \quad \text{and}$$

$$\mathrm{Re}\left(\frac{t}{s} + 1\right) \geq m \frac{|M - \bar{a}e^{i\theta}|^2}{M^2 - |a|^2},$$

where $\theta \in \mathbb{R}$ and

$$m \geq n + \frac{M^2 - |a|^2 - M\beta}{M^2 - |a|^2 + M\beta}.$$

Thus, the class $\Psi_{n,\beta}(\Omega, M, a)$ consists of those functions $\psi : \mathbb{C}^3 \to \mathbb{C}$ that are continuous in a domain $D \subset \mathbb{C}^3$ with $(a, 0, 0) \in D$ and $\psi(a, 0, 0) \in \Omega$, and satisfying the β-admissibility condition (3.2).

   If $a = 0$, then (3.2) simplifies to

$$\psi(r, s, t) \notin \Omega \quad \text{whenever} \quad (r, s, t) \in D, \ r = M e^{i\theta}, \ s = mM e^{i\theta}, \ \text{and}$$

$$\mathrm{Re}\left(\frac{t}{s} + 1\right) \geq m,$$

where $\theta \in \mathbb{R}$ and

$$m \geq n + \frac{M - \beta}{M + \beta}.$$

Equivalently, the condition is
(3.3)

$$\psi(Me^{i\theta}, Ke^{i\theta}, L) \notin \Omega \quad \text{whenever} \quad (Me^{i\theta}, Ke^{i\theta}, L) \in D,$$

$$K \geq \left(n + \frac{M - \beta}{M + \beta}\right) M, \quad \text{and} \quad \operatorname{Re}(Le^{-i\theta}) \geq \left(n - \frac{2\beta}{M + \beta}\right) K,$$

where $\theta \in \mathbb{R}$ and $n \geq 1$.

In this particular case, Theorem 3.1 can be expressed in the following form:

Theorem 3.3. *Let* $p \in \mathcal{H}_\beta[a, n]$ *with* $|a| < M$, $0 < \beta \leq (M^2 - |a|^2)/M$, $M > 0$.

(i) *Let* $\psi \in \Psi_{n,\beta}(\Omega, M, a)$ *with associated domain* D. *If* $(p(z), zp'(z), z^2p''(z)) \in D$ *and*
$$\psi(p(z), zp'(z), z^2p''(z)) \in \Omega \quad (z \in \mathbb{D}),$$
then $|p(z)| < M$.

(ii) *Let* $\psi \in \Psi_{n,\beta}(M, a)$ *with associated domain* D. *If* $(p(z), zp'(z), z^2p''(z)) \in D$ *and*
$$|\psi(p(z), zp'(z), z^2p''(z))| < M \quad (z \in \mathbb{D}),$$
then $|p(z)| < M$.

Case 2. The half-plane $\Delta = \{w : \operatorname{Re} w > 0\}$. Here the function

$$q(z) = \frac{a + \bar{a}z}{1 - z} \quad (z \in \mathbb{D}),$$

where $\operatorname{Re} a > 0$, is univalent in $\overline{\mathbb{D}} \setminus \{1\}$ and satisfies $q(\mathbb{D}) = \Delta$, $q(0) = a$ and $q \in Q$. Let $\Psi_{n,\beta}(\Omega, a) := \Psi_{n,\beta}(\Omega, q)$ and when $\Omega = \Delta$, denote the class by $\Psi_{n,\beta}(a)$. Comparing with Lemma 2.2f in [15, p. 26], the condition of β-admissibility becomes

$$\psi(r, s, t) \notin \Omega \quad \text{whenever} \quad (r, s, t) \in D,$$

$$r = q(\zeta) = i\rho,$$

$$s = m\zeta q'(\zeta) = -\frac{m}{2}\frac{|a - i\rho|^2}{\operatorname{Re} a}, \quad \text{and}$$

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq 0,$$

where $\rho \in \mathbb{R}$ and

$$m \geq n + \frac{2\operatorname{Re} a - \beta}{2\operatorname{Re} a + \beta}.$$

Equivalently,

$$\psi(i\rho, \sigma, \mu + i\nu) \notin \Omega \quad \text{whenever} \quad (i\rho, \sigma, \mu + i\nu) \in D,$$

(3.4)
$$\sigma \leq -\frac{1}{2}\left(n + \frac{2\operatorname{Re} a - \beta}{2\operatorname{Re} a + \beta}\right)\frac{|a - i\rho|^2}{\operatorname{Re} a}, \quad \text{and}$$

$$\sigma + \mu \leq 0,$$

where $\rho, \sigma, \mu, \nu \in \mathbb{R}$ and $n \geq 1$. Thus, the class $\Psi_{n,\beta}(\Omega, a)$ consists of those functions $\psi : \mathbb{C}^3 \to \mathbb{C}$ that are continuous in a domain $D \subset \mathbb{C}^3$ with $(a, 0, 0) \in D$ and $\psi(a, 0, 0) \in \Omega$, satisfying the β-admissibility condition (3.4).

If $a = 1$, then (3.4) simplifies to

$$\psi(i\rho, \sigma, \mu + i\nu) \notin \Omega \quad \text{whenever} \quad (i\rho, \sigma, \mu + i\nu) \in D,$$

(3.5)
$$\sigma \leq -\frac{1}{2}\left(n + \frac{2 - \beta}{2 + \beta}\right)(1 + \rho^2), \quad \text{and}$$

$$\sigma + \mu \leq 0,$$

where $\rho, \sigma, \mu, \nu \in \mathbb{R}$, and $n \geq 1$, a condition much easier to check.

In this particular case, Theorem 3.1 can be rephrased in the following form.

Theorem 3.4. *Let* $p \in \mathcal{H}_\beta[a, n]$ *with* $\operatorname{Re} a > 0$, $0 < \beta \leq 2\operatorname{Re} a$.

(i) *Let* $\psi \in \Psi_{n,\beta}(\Omega, a)$ *with associated domain* D. *If* $(p(z), zp'(z), z^2 p''(z)) \in D$ *and*

$$\psi(p(z), zp'(z), z^2 p''(z)) \in \Omega \quad (z \in \mathbb{D}),$$

then $\operatorname{Re} p(z) > 0$.

(ii) *Let* $\psi \in \Psi_{n,\beta}(a)$ *with associated domain* D. *If* $(p(z), zp'(z), z^2 p''(z)) \in D$ *and*

$$\operatorname{Re} \psi(p(z), zp'(z), z^2 p''(z)) > 0 \quad (z \in \mathbb{D}),$$

then $\operatorname{Re} p(z) > 0$.

4. Examples

In this section, examples of differential inequalities and subordinations are presented to obtain several interesting results. These are applications of β-admissible functions ψ in $\Psi_{n,\beta}(\Omega, q)$, by judicious choices of ψ. For the sake of comparison, we shall look at several examples that were considered by Miller and Mocanu in [15, pp. 36–42].

The first example involves a disk of radius M and is an application of Theorem 3.3.

Example 4.1. Let $\psi(r, s, t) = r + s + t$, $a = 0$, and $\Omega = h(\mathbb{D})$, where

$$h(z) = \left(\left(n + \frac{M - \beta}{M + \beta}\right)^2 + 1\right) Mz.$$

To apply Theorem 3.3, we need to show that $\psi \in \Psi_{n,\beta}(\Omega, M, 0)$ for $n \geq 1$ and $\beta \leq M$. The function ψ evidently satisfies the first two admissibility conditions: ψ is continuous in the domain $D = \mathbb{C}^3$, $(0, 0, 0) \in D$ and $\psi(0, 0, 0) = 0 \in \Omega$. It remains to show the β-admissibility condition (3.3) is satisfied. Since

$$\psi(Me^{i\theta}, Ke^{i\theta}, L) = Me^{i\theta} + Ke^{i\theta} + L,$$

then

$$|\psi(Me^{i\theta}, Ke^{i\theta}, L)| = |M + K + Le^{-i\theta}|$$

$$\geq M + K + \operatorname{Re}(Le^{-i\theta})$$

$$\geq M + K + \left(n - \frac{2\beta}{M + \beta}\right) K$$

$$= M + \left(n + \frac{M - \beta}{M + \beta}\right) K$$

$$\geq M + \left(n + \frac{M - \beta}{M + \beta} \right)^2 M$$

$$= \left(\left(n + \frac{M - \beta}{M + \beta} \right)^2 + 1 \right) M,$$

whenever $K \geq (n + (M - \beta)/(M + \beta))M$, $\mathrm{Re}(Le^{-i\theta}) \geq (n - 2\beta/(M + \beta))K$, $\theta \in \mathbb{R}$, and $n \geq 1$. Thus, $\psi \in \Psi_{n,\beta}(\Omega, M, 0)$ for $n \geq 1$. Theorem 3.3 now yields the following differential subordination result:

Let $p \in \mathcal{H}_\beta[0, n]$ with $0 < \beta \leq M$. If

$$|p(z) + zp'(z) + z^2 p''(z)| < \left(\left(n + \frac{M - \beta}{M + \beta} \right)^2 + 1 \right) M \quad (z \in \mathbb{D}),$$

then $|p(z)| < M$.

Remark 4.1. If $\beta = M$, then Example 4.1 reduces to Example 2.4a in [15, p. 36]. Since

$$(n^2 + 1)M \leq \left(\left(n + \frac{M - \beta}{M + \beta} \right)^2 + 1 \right) M \quad \text{for } \beta \leq M,$$

it is clear that Example 4.1 extends Example 2.4a in [15, p. 36] for functions $p \in \mathcal{H}_\beta[0, n]$.

The next example involves a function with positive real part and is an application of Theorem 3.4.

Example 4.2. Let $\psi(r, s, t) = 1 - r^2 + 5s + t$, $a = 1$ and

$$\Omega = \left\{ w : |w| < \frac{6 - \beta}{2 + \beta} \right\}.$$

Now Theorem 3.4 is applicable provided $\psi \in \Psi_{n,\beta}(\Omega, 1)$ for $n \geq 1$ and $\beta \leq 2$. The function ψ is continuous in the domain $D = \mathbb{C}^3$, $(1, 0, 0) \in D$ and $\psi(1, 0, 0) \in \Omega$. To show that the β-admissibility condition (3.5) is satisfied, consider $\psi(i\rho, \sigma, \mu + i\nu) = 1 + \rho^2 + 5\sigma + \mu + i\nu$. Then

$$|\psi(i\rho, \sigma, \mu + i\nu)| = |1 + \rho^2 + 5\sigma + \mu + i\nu|$$

$$\geq -(1 + \rho^2 + 5\sigma + \mu)$$

$$\geq -(1 + \rho^2) - 4\sigma$$

$$\geq -(1 + \rho^2) + 2 \left(n + \frac{2 - \beta}{2 + \beta} \right)(1 + \rho^2)$$

$$\geq 2 \left(n + \frac{2 - \beta}{2 + \beta} \right) - 1$$

$$\geq 2 \left(1 + \frac{2 - \beta}{2 + \beta} \right) - 1$$

$$= \frac{6 - \beta}{2 + \beta}$$

whenever $\rho \in \mathbb{R}$, $\sigma \leq -\left(n + (2 - \beta)/(2 + |\beta|)\right)(1 + \rho^2)/2$, $\sigma + \mu \leq 0$ and $n \geq 1$. Thus, $\psi \in \Psi_{n,\beta}(\Omega, 1)$, and Theorem 3.4 yields the following differential inequality result:

Let $p \in \mathcal{H}_\beta[1, n]$, and $0 < \beta \leq 2$. If

$$|z^2 p''(z) + 5zp'(z) - p^2(z) + 1| < \frac{6 - \beta}{2 + \beta} \qquad (z \in \mathbb{D}),$$

then $\operatorname{Re} p(z) > 0$.

Remark 4.2. If $\beta = 2$, then Example 4.2 reduces to Example 2.4i in [15, p. 41]. Since

$$\frac{6 - \beta}{2 + \beta} \geq 1 \quad \text{for} \ \ \beta \leq 2,$$

Example 4.2 extends Example 2.4i in [15, p. 41].

The next two examples illustrate the sensitivity of the class of β-admissible functions to the value n.

Example 4.3. Let $\psi(r, s, t) = r + s + 1 - r^2$, $\Omega = \Delta = \{w : \operatorname{Re} w > 0\}$ and $a = 1$. We first show that $\psi \in \Psi_{n,\beta}(1)$ for $n \geq (2 + 3\beta)/(2 + \beta)$ and $\beta \leq 2$. The function ψ is continuous in the domain $D = \mathbb{C}^3$, $(1, 0, 0) \in D$, and $\operatorname{Re} \psi(1, 0, 0) = 1 > 0$ so that $\psi(1, 0, 0) \in \Omega$. To verify the β-admissibility condition (3.5) is satisfied, consider

$$\psi(i\rho, \sigma, \mu + i\nu) = i\rho + \sigma + 1 + \rho^2.$$

Then

$$\operatorname{Re} \psi(i\rho, \sigma, \mu + i\nu) = \sigma + 1 + \rho^2$$

$$\leq -\frac{1}{2}\left(n + \frac{2 - \beta}{2 + \beta}\right)(1 + \rho^2) + 1 + \rho^2$$

$$= (1 + \rho^2)\left[-\frac{n}{2} + \frac{2 + 3\beta}{2(2 + \beta)}\right]$$

$$= \frac{1}{2}(1 + \rho^2)\left[\frac{2 + 3\beta}{2 + \beta} - n\right] \leq 0,$$

whenever $\rho \in \mathbb{R}$, $\sigma \leq -[n + (2 - \beta)/(2 + \beta)](1 + \rho^2)/2$, and $n \geq (2 + 3\beta)/(2 + \beta)$. Thus, $\psi \in \Psi_{n,\beta}(1)$ for $n \geq (2 + 3\beta)/(2 + \beta)$ and $\beta \leq 2$. Therefore, by Theorem 3.4, the following differential inequality result is obtained:

Let $p \in \mathcal{H}_\beta[1, n]$ with $0 < \beta \leq 2$ and $n \geq (2 + 3\beta)/(2 + \beta)$. If

$$\operatorname{Re}\left(p(z) + zp'(z) + 1 - p^2(z)\right) > 0 \quad (z \in \mathbb{D}),$$

then $\operatorname{Re} p(z) > 0$.

Remark 4.3. If $\beta = 2$, then

$$\frac{2 + 3\beta}{2 + \beta} = 2$$

so that Example 4.3 reduces to Example 2.4g in [15, p. 40]. For $\beta \leq 2$, then

$$\frac{2+3\beta}{2+\beta} \leq 2,$$

and Example 4.3 extends Example 2.4g in [15, p. 40].

The next example has a similar restriction.

Example 4.4. Let $\psi(r,s,t) = 2 - r^2 + 3s + t$, $\Omega = \Delta = \{w : \operatorname{Re} w > 0\}$ and $a = 1$. The function ψ is continuous in the domain $D = \mathbb{C}^3$, $(1,0,0) \in D$, and $\operatorname{Re} \psi(1,0,0) = 1 > 0$ so that $\psi(1,0,0) \in \Omega$. To show the β-admissibility condition (3.5) is satisfied, consider

$$\psi(i\rho, \sigma, \mu + i\nu) = 2 + \rho^2 + 3\sigma + \mu + i\nu.$$

Then

$$\operatorname{Re}\psi(i\rho, \sigma, \mu + i\nu) = 2 + \rho^2 + 2\sigma + (\sigma + \mu)$$
$$\leq 2 + \rho^2 + 2\sigma$$
$$= \left(2 - n - \frac{2-\beta}{2+\beta}\right) + \left(1 - n - \frac{2-\beta}{2+\beta}\right)\rho^2$$
$$= \left(\frac{2+3\beta}{2+\beta} - n\right) + \left(\frac{2\beta}{2+\beta} - n\right)\rho^2 \leq 0,$$

whenever $\rho \in \mathbb{R}$, $\sigma \leq -(n + (2-\beta)/(2+\beta))(1 + \rho^2)/2$, $\sigma + \mu \leq 0$ and $n \geq (2+3\beta)/(2+\beta)$. Thus, $\psi \in \Psi_{n,\beta}(1)$ for $n \geq (2+3\beta)/(2+\beta)$ and $\beta \leq 2$. Therefore, Theorem 3.4 yields the following differential inequality result:

Let $p \in \mathcal{H}_\beta[1,n]$, $0 < \beta \leq 2$ and $n \geq (2+3\beta)/(2+\beta)$. If
$$\operatorname{Re}\left(z^2 p''(z) + 3z p'(z) - p^2(z) + 2\right) > 0 \quad (z \in \mathbb{D}),$$
then $\operatorname{Re} p(z) > 0$.

Remark 4.4. If $\beta = 2$, then $n \geq (2+3\beta)/(2+\beta) = 2$ so that Example 4.4 extends Example 2.4h in [15, p. 40].

Example 4.5. In this example, consider the class $\Psi_{2,\beta}(\Omega, q)$, where $q(z) = 1 + z$, $\beta \leq 1$, and
$$\Omega = \left\{w : |w| < \frac{4(3+2\beta)}{(1+\beta)^2}\right\}.$$
The function $\psi(r,s,t) = 1 - r^2 + 3s + t$ is continuous in the domain $D = \mathbb{C}^3$, $(1,0,0) \in D$ and $\psi(1,0,0) \in \Omega$. It remains to show that the β-admissibility condition is satisfied. If we set $r_0 = q(\zeta)$, $s_0 = m\zeta q'(\zeta)$ and
$$\operatorname{Re}\left(\frac{t_0}{s_0} + 1\right) \geq m \operatorname{Re}\left(\zeta \frac{q''(\zeta)}{q'(\zeta)} + 1\right),$$
where $|\zeta| = 1$ and $m \geq 2 + (1-\beta)/(1+\beta)$, then
$$r_0 = 1 + \zeta, \quad s_0 = m\zeta, \quad \text{and} \quad \operatorname{Re}(t_0\bar{\zeta}) \geq m(m-1).$$

Since $\psi(r_0, s_0, t_0) = (3m - 2)\zeta - \zeta^2 + t_0$, it follows that

$$|\psi(r_0, s_0, t_0)| = |3m - 2 - \zeta + t_0\bar{\zeta}|$$
$$\geq 3m - 2 - \operatorname{Re}\zeta + \operatorname{Re}(t_0\bar{\zeta})$$
$$\geq 3m - 3 + m(m - 1)$$
$$= (m + 3)(m - 1)$$
$$\geq \left(5 + \frac{1 - \beta}{1 + \beta}\right)\left(1 + \frac{1 - \beta}{1 + \beta}\right)$$
$$= \frac{4(3 + 2\beta)}{(1 + \beta)^2}.$$

Thus $\psi(r_0, s_0, t_0) \notin \Omega$, and the β-admissibility condition is satisfied, that is, $\psi \in \Psi_{2,\beta}(\Omega, q)$. Theorem 3.1 now yields the following :

Let $p \in \mathcal{H}_\beta[1, 2]$ and $0 < \beta \leq 1$. If

$$|z^2 p''(z) + 3z p'(z) - p^2(z) + 1| < \frac{4(3 + 2\beta)}{(1 + \beta)^2} \quad (z \in \mathbb{D}),$$

then

$$|p(z) - 1| < 1 \quad (z \in \mathbb{D}).$$

Remark 4.5. If $\beta = 1$, then
$$\frac{4(3 + 2\beta)}{(1 + \beta)^2} = 5$$
so that Example 4.5 extends Example 2.4k in [15, p. 42]. The assumption $p \in \mathcal{H}_\beta[1, 2]$ for $\beta \leq 1$ implies that
$$\frac{4(3 + 2\beta)}{(1 + \beta)^2} \geq 5.$$

5. Applications in univalent function theory

This section looks at several interesting applications of the theory developed in the earlier sections to normalized convex and starlike univalent functions with fixed second coefficient. The results obtained here extend those given in Section 2.6 of [15, p. 56].

Let $\mathcal{A}_n$ be the class consisting of analytic functions f defined in $\mathbb{D}$ of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$, and $\mathcal{A} := \mathcal{A}_1$. The class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, $0 \leq \alpha < 1$, consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

Similarly, the class $\mathcal{C}(\alpha)$ of convex functions of order α, $0 \leq \alpha < 1$, consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

When $\alpha = 0$, these classes are respectively denoted by $\mathcal{S}^*$ and $\mathcal{C}$. Let $\mathcal{A}_{n,b}$ denote the class of functions $f \in \mathcal{A}_n$ of the form

$$f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots,$$

with fixed coefficient $a_{n+1} = b$. We write $\mathcal{A}_{1,b}$ as $\mathcal{A}_b$.

Theorem 5.1. *If $f(z) = z + a_2 z^2 + \ldots \in \mathcal{C}$, then $f \in \mathcal{S}^*(\alpha)$ where α is the smallest positive root of the equation*

(5.1) $$2\alpha^3 - \alpha^2 |a_2| - 4\alpha + 2 = 0$$

in the interval $[1/2, 2/3]$.

Proof. The function g defined by

$$g(\alpha) = 2\alpha^3 - \alpha^2 |a_2| - 4\alpha + 2$$

is continuous in $[1/2, 2/3]$. Let $b := a_2$. Since $f \in \mathcal{C}$, $|b| \leq 1$ and g satisfies

$$g\left(\frac{1}{2}\right) = \frac{1}{4}(1 - |b|) \geq 0, \quad \text{and} \quad g\left(\frac{2}{3}\right) = -\frac{2}{27}(1 + 6|b|) \leq 0.$$

Therefore there exists a root of $g(\alpha) = 0$ in $[1/2, 2/3]$. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

(5.2) $$p(z) := \frac{zf'(z)}{f(z)} - \alpha,$$

where α is the smallest positive root of (5.1). Since f is a convex univalent function in $\mathcal{A}_b$, the function

$$p(z) = (1 - \alpha) + bz + \cdots$$

is analytic in $\mathbb{D}$. Thus $p \in \mathcal{H}_b[1 - \alpha, 1]$, and $\alpha \leq 2/3 < 1$ readily yields

$$p(0) = 1 - \alpha > 0.$$

From (5.2), it follows that

$$\frac{zf'(z)}{f(z)} = p(z) + \alpha$$

so that

(5.3) $$\frac{zf''(z)}{f'(z)} + 1 = p(z) + \alpha + \frac{zp'(z)}{p(z) + \alpha} = \psi(p(z), zp'(z)),$$

where

$$\psi(r, s) = r + \alpha + \frac{s}{r + \alpha}.$$

The function ψ is continuous in the domain $D = (\mathbb{C} \setminus \{-\alpha\}) \times \mathbb{C}$, $(1 - \alpha, 0) \in D$, and

$$\mathrm{Re}\, \psi(1 - \alpha, 0) = 1 > 0.$$

We now show that the β-admissibility condition (3.4) is satisfied. Since

$$\psi(i\rho, \sigma) = i\rho + \alpha + \frac{\sigma}{\alpha^2 + \rho^2}(\alpha - i\rho),$$

it follows that

$$\mathrm{Re}\, \psi(i\rho, \sigma) = \alpha + \frac{\alpha\sigma}{\alpha^2 + \rho^2}$$

$$\leq \alpha - \frac{1}{2}\frac{\alpha}{\alpha^2 + \rho^2}\left(1 + \frac{2(1-\alpha) - |b|}{2(1-\alpha) + |b|}\right)\frac{(1-\alpha)^2 + \rho^2}{1 - \alpha}$$

$$= \alpha - \frac{2\alpha}{2(1-\alpha) + |b|}\frac{(1-\alpha)^2 + \rho^2}{\alpha^2 + \rho^2}.$$

Since the function h given by

$$h(t) = \frac{(1-\alpha)^2 + t}{\alpha^2 + t}, \quad t \geq 0,$$

is an increasing function of t for $\alpha \geq 1/2$, clearly

$$\frac{(1-\alpha)^2 + \rho^2}{\alpha^2 + \rho^2} \geq \frac{(1-\alpha)^2}{\alpha^2}.$$

Thus, it follows from (5.1) that

$$\operatorname{Re}\psi(i\rho, \sigma) \leq \alpha - \frac{2\alpha}{2(1-\alpha) + |b|}\frac{(1-\alpha)^2}{\alpha^2}$$

$$= \frac{\alpha^2|b| - 2\alpha^3 + 4\alpha - 2}{\alpha[2(1-\alpha) + |b|]} = 0,$$

whenever $\rho \in \mathbb{R}$ and

$$\sigma \leq -\frac{1}{2}\left(1 + \frac{2\operatorname{Re}p(0) - |b|}{2\operatorname{Re}p(0) + |b|}\right)\frac{|p(0) - i\rho|^2}{\operatorname{Re}p(0)}, \quad p(0) = 1 - \alpha.$$

Thus $\psi \in \Psi_b(1 - \alpha)$. The hypothesis and (5.3) give

$$\operatorname{Re}\psi(p(z), zp'(z)) > 0 \quad (z \in \mathbb{D}).$$

Therefore, Theorem 3.4 (ii) shows that p satisfies $\operatorname{Re}p(z) > 0$, and thus $f \in \mathcal{S}^*(\alpha)$. ∎

Remark 5.1. The roots of (5.1) in $[1/2, 2/3]$ are decreasing as a function of $|a_2|$, $|a_2| \in (0, 1]$. If $|b| = |a_2| = 1$, then (5.1) becomes

$$2\alpha^3 - \alpha^2 - 4\alpha + 2 = (\alpha^2 - 2)(2\alpha - 1) = 0.$$

Thus $\alpha = 1/2$. Therefore, Theorem 5.1 reduces to Theorem 2.6a in [15, p. 57] in this case, and provides an improvement of the Marx-Strohhäcker's result that $f \in \mathcal{C}$ implies $f \in \mathcal{S}^*(1/2)$.

Theorem 5.2. *If $f = z + a_2 z^2 + \ldots \in \mathcal{C}$, then*

$$\operatorname{Re}\sqrt{f'(z)} > \alpha,$$

where α is given by

(5.4) $$\alpha = \frac{10 + |a_2| - \sqrt{|a_2|^2 + 20|a_2| + 4}}{12}.$$

Proof. Let $b =: a_2$. First note that α given by (5.4) satisfies the equation

(5.5) $$6\alpha^2 - (10 + |b|)\alpha + 4 = 0.$$

If $\alpha < 1/2$, then

$$10 + |b| - \sqrt{|b|^2 + 20|b| + 4} < 6$$

and leads to $|b| > 1$, which contradicts f being convex. Similarly, if $\alpha > 2/3$, then $|b| < 0$ which is again a contradiction. Thus $\alpha \in [1/2, 2/3]$.

Define the function $p : \mathbb{D} \to \mathbb{C}$ by

(5.6)
$$p(z) := \sqrt{f'(z)} - \alpha.$$

Since $f \in \mathcal{A}_b$ and f is convex univalent, the function

$$p(z) = (1 - \alpha) + bz + \cdots$$

is analytic in $\mathbb{D}$. Thus $p \in \mathcal{H}_b[1 - \alpha, 1]$, and because $\alpha \leq 2/3 < 1$, then

$$p(0) = 1 - \alpha > 0.$$

Now (5.6) yields

$$f'(z) = (p(z) + \alpha)^2$$

so that

(5.7)
$$\frac{z f''(z)}{f'(z)} + 1 = \frac{2z p'(z)}{p(z) + \alpha} + 1 = \psi(p(z), z p'(z)),$$

where

$$\psi(r, s) = \frac{2s}{r + \alpha} + 1.$$

The function ψ is continuous in the domain $D = (\mathbb{C} \setminus \{-\alpha\}) \times \mathbb{C}$, $(1 - \alpha, 0) \in D$ and $\operatorname{Re} \psi(1 - \alpha, 0) = 1 > 0$. To verify the β-admissibility condition (3.4) is satisfied, consider

$$\psi(i\rho, \sigma) = \frac{2\sigma}{\alpha^2 + \rho^2}(\alpha - i\rho) + 1.$$

Then

$$\operatorname{Re} \psi(i\rho, \sigma) = \frac{2\alpha\sigma}{\alpha^2 + \rho^2} + 1$$

$$\leq -\frac{\alpha}{\alpha^2 + \rho^2}\left(1 + \frac{2(1 - \alpha) - |b|}{2(1 - \alpha) + |b|}\right)\frac{(1 - \alpha)^2 + \rho^2}{1 - \alpha} + 1$$

$$= -\frac{4\alpha}{2(1 - \alpha) + |b|}\frac{(1 - \alpha)^2 + \rho^2}{\alpha^2 + \rho^2} + 1.$$

Using (5.5) and the monotonicity of the function

$$h(t) = \frac{(1 - \alpha)^2 + t}{\alpha^2 + t}, \quad t \geq 0,$$

it follows that

$$\operatorname{Re} \psi(i\rho, \sigma) \leq -\frac{4\alpha}{2(1 - \alpha) + |b|}\frac{(1 - \alpha)^2}{\alpha^2} + 1$$

$$= \frac{(10 + |b|)\alpha - 6\alpha^2 - 4}{\alpha[2(1 - \alpha) + |b|]} = 0,$$

whenever $\rho \in \mathbb{R}$ and

$$\sigma \leq -\frac{1}{2}\left(1 + \frac{2\operatorname{Re} p(0) - |b|}{2\operatorname{Re} p(0) + |b|}\right)\frac{|p(0) - i\rho|^2}{\operatorname{Re} p(0)}, \quad p(0) = 1 - \alpha.$$

Thus $\psi \in \Psi_b(1 - \alpha)$.

The hypothesis and (5.7) yield

$$\operatorname{Re} \psi(p(z), z p'(z)) > 0 \quad (z \in \mathbb{D}).$$

R. M. Ali, S. Nagpal and V. Ravichandran

Therefore, we conclude from Theorem 3.4(ii) that p satisfies

$$\operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{D}).$$

This is equivalent to

$$\operatorname{Re} \sqrt{f'(z)} > \frac{10 + |b| - \sqrt{|b|^2 + 20|b| + 4}}{12}, \quad b = a_2. \qquad \blacksquare$$

Remark 5.2. The roots of (5.4) are decreasing as a function of $|a_2|$, $0 < |a_2| \le 1$. If $|a_2| = 1$, then α given by (5.4) reduces to $1/2$. Therefore, Theorem 5.2 improves Theorem 2.6a in [15, p. 57].

Theorem 5.3. *If $f = z + a_2 z^2 + \ldots \in \mathcal{S}^*$, then*

$$\operatorname{Re} \sqrt{\frac{f(z)}{z}} > \alpha,$$

where α is given by

(5.8) $$\alpha = \frac{20 + |a_2| - \sqrt{16 + |a_2|^2 + 40|a_2|}}{24}.$$

Proof. The result follows from Theorem 5.2 and the fact that the classes of convex and starlike functions functions are related by the Alexander relation $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. $\blacksquare$

Remark 5.3. If $|a_2| = 2$ then α given by (5.8) reduces to $1/2$. Therefore, Theorem 5.3 improves Theorem 2.6e in [15, p. 62].

Acknowledgement. The work presented here was supported in parts by FRGS and RU grants from Universiti Sains Malaysia. The authors are thankful to the referees for their useful comments.

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